Signal Analysis & Communication ECE 355

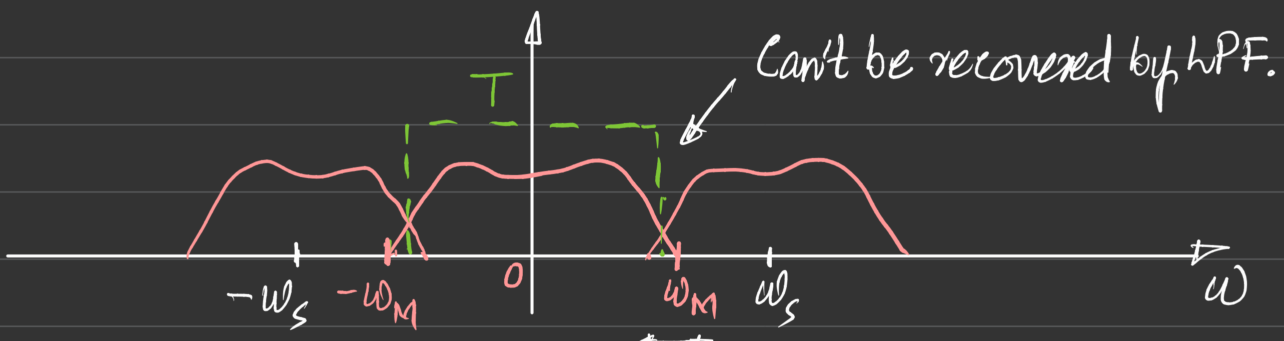
Ch. 7.3. Undersampling/Aliasing

Lecture 30

23-11-2023

Ch.7.3 UNDER SAMPLING / ALIASING

- This is the case when $\omega_s < 2\omega_M$
- Here, the spectrum of $x_{(t)}$, $X_{(j\omega)}$, is no longer replicated in $X_{p(j\omega)}$ & thus is no longer recoverable by low pass filtering.



$$X_{r(j\omega)} = X_{p(j\omega)} H_{(j\omega)}$$

where

$$H_{(j\omega)} = \begin{cases} T, & |\omega| < \omega_s/2 \\ 0, & \text{otherwise.} \end{cases} \quad \text{— } 1$$

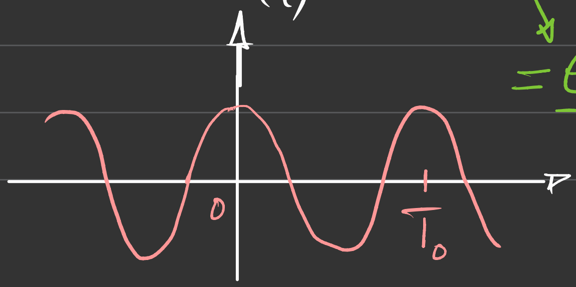
$$\Rightarrow X_{r(j\omega)} \neq X_{(j\omega)}$$

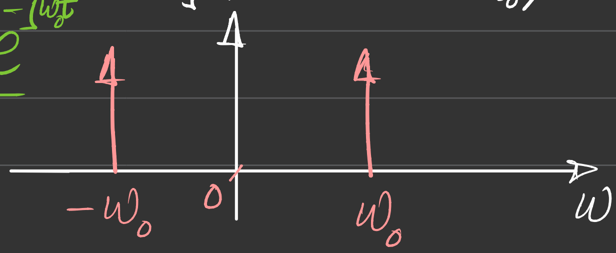
- To have more insights of 'Aliasing' effect, let's look at an example.

- Example

$$x_{(t)} = \cos\omega_0 t \xleftrightarrow{\mathcal{F}} X_{(j\omega)} = \pi(\delta_{(\omega-\omega_0)} + \delta_{(\omega+\omega_0)})$$

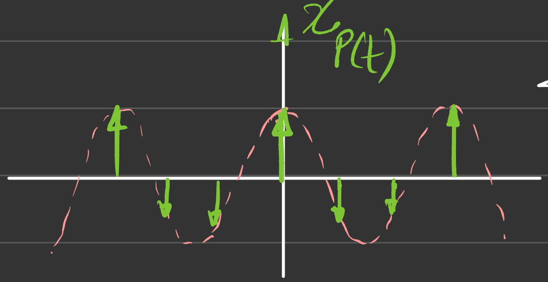
$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2}$$

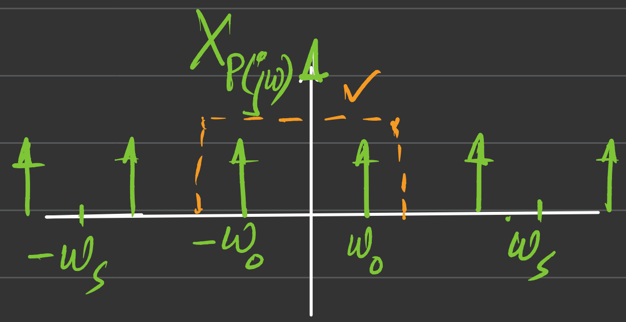




Case I $\omega_s = 3\omega_0$, $T = \frac{2\pi}{\omega_s} = \frac{T_0}{3}$ (3 samples in each period)

$\boxed{\omega_s > 2\omega_0}$

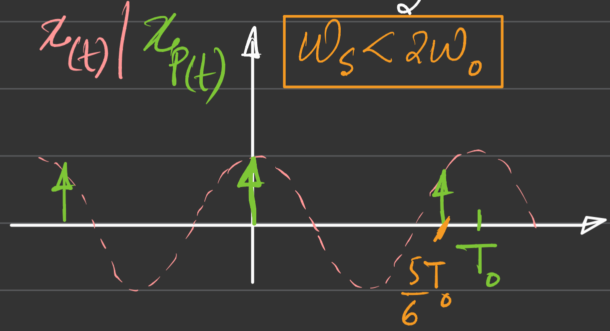




$x_{r(t)} = \cos\omega_0 t$ ✓ we can reconstruct the sig. as of original CT.

Case II $\omega_s = \frac{6}{5}\omega_0$, $T = \frac{2\pi}{\omega_s} = \frac{2\pi}{6/5\,\omega_0} = \frac{5}{6}T_0$

$\boxed{\omega_s < 2\omega_0}$



$-\omega_s + \omega_0$
$-\frac{6}{5}\omega_0 + \omega_0$
$= -\frac{\omega_0}{5}$

$\omega_s - \omega_0 = \frac{6}{5}\omega_0 - \omega_0$
$= \frac{6\omega_0 - 5\omega_0}{5}$
$= \frac{\omega_0}{5}$

$x_{r(t)} = \cos\left(\frac{\omega_0}{5}t\right)$

$\neq x_{(t)}$

$\frac{1}{2}\left(e^{\frac{j\omega_0 t}{5}} + e^{-\frac{j\omega_0 t}{5}}\right)$

$\Rightarrow X_{r(j\omega)} = \pi\left[\delta\left(\omega - \frac{\omega_0}{5}\right) + \delta\left(\omega + \frac{\omega_0}{5}\right)\right]$

$-180°$ phase change.

That's Aliasing.

NOTE 1

- Even with "Aliasing", $x_{r}(nT) = x(nT)$ for any $n \in \mathbb{Z}$

  although $x_r(t) \neq x(t)$

- Let's look at the effect of undersampling more closely by looking at the filtered sig. (interpolated) spectra $X_r(j\omega)$, & the corresponding $x_r(t)$.

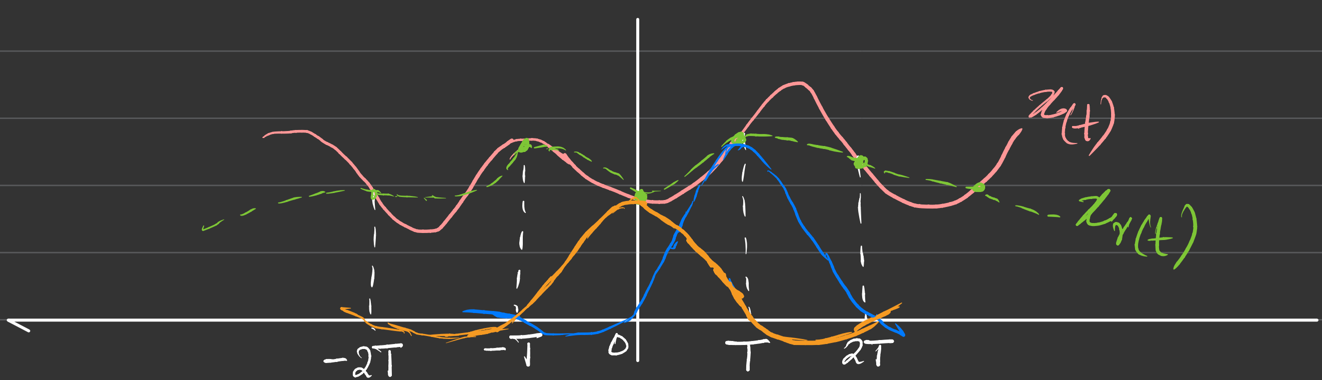
$$X_r(j\omega) = X_p(j\omega)\, H(j\omega)$$

$$x_r(t) = \sum_{k=-\infty}^{\infty} x(kT) \frac{\sin\left(\frac{\pi}{T}(t-kT)\right)}{\frac{\pi}{T}(t-kT)}$$

$\times$ in freq. $\Updownarrow$ $*$ in time

- The sampling is done at $t=nT$. At $t=nT$,

$$\frac{\sin\left(\frac{\pi}{T}(nT-kT)\right)}{\frac{\pi}{T}(nT-kT)} = \frac{\sin((n-k)\pi)}{(n-k)\pi} = \begin{cases} 1, & n=k \\ 0, & n\neq k \end{cases}$$



- $x_r(t)$ is the "smoothest" sig. that goes through the points $x(0), x(T), x(2T), \ldots$

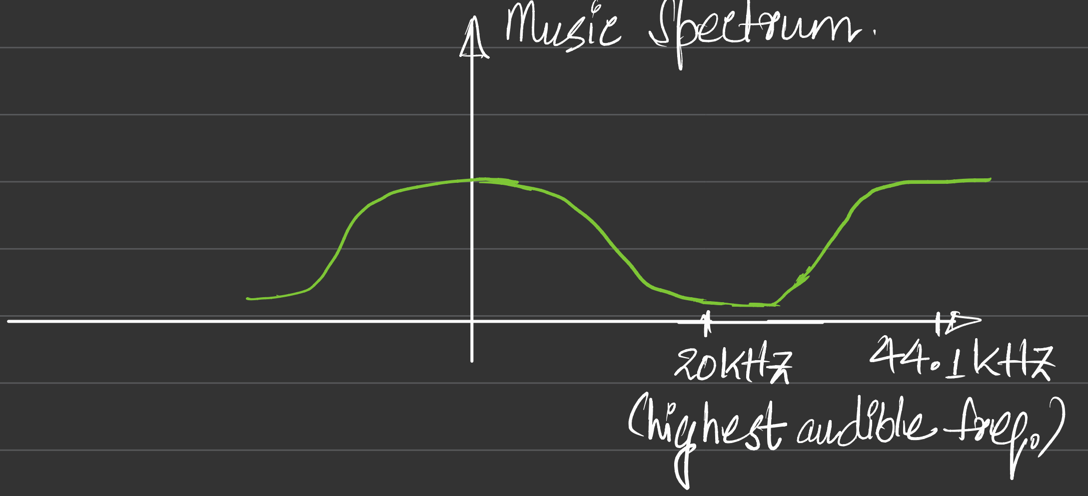
- More samples with $\omega_s > 2\omega_M$ would be needed to reconstruct $x(t)$ here.
  (More frequent sinc pulses)

NOTE 2:

- Aliasing is unavoidable in practice.
- The band-limited sig. cannot also be time limited.
- This means sig. would need infinite support in time - impractical.
- To avoid aliasing in that case, there are some practical solutions:

① Oversampling $\omega_s > 2\omega_M$ (not use equality)

Eg.

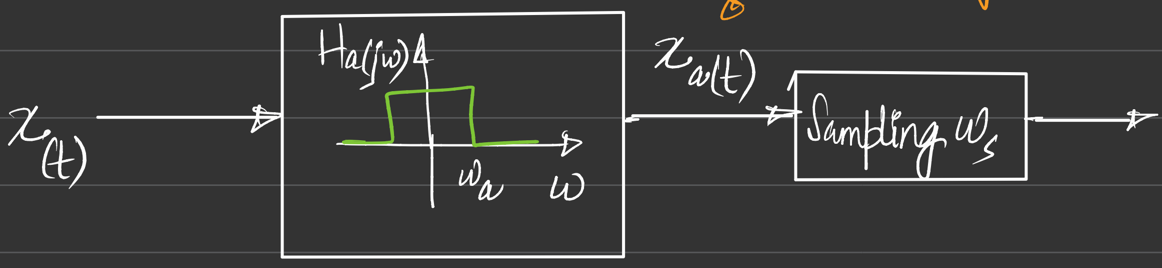


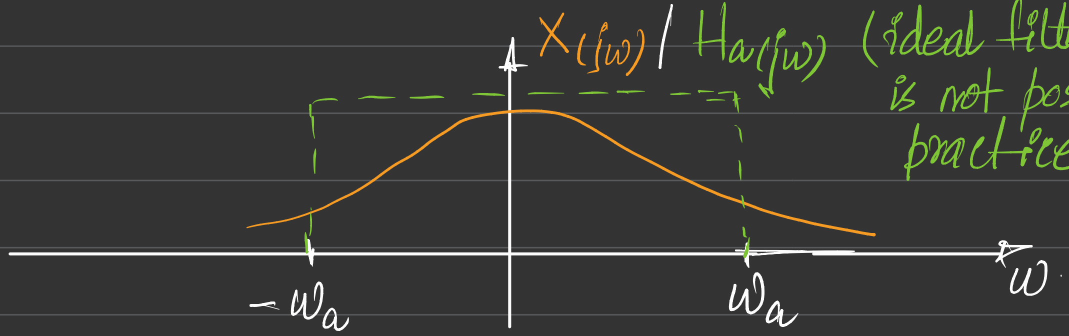
$\omega_s > 2\omega_M$

Some professionals use: 48 kHz
96 kHz

② Anti-Aliasing Filters

Limit sig. BW.





(ideal filter, which again is not possible in practice)